

D-Strings on D-Manifolds

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We study the mechanism for appearance of massless solitons in type II string compactifications. We find that by combining T -duality with strong/weak duality of type IIB in 10 dimensions enhanced gauge symmetries and massless solitonic hypermultiplets encountered in Calabi-Yau compactifications can be studied perturbatively using D-strings (the strong/weak dual to type IIB string) compactified on “D-manifolds”. In particular the nearly massless solitonic states of the type IIB compactifications correspond to elementary states of D-strings. As examples we consider the D-string description of enhanced gauge symmetries for type IIA string compactification on ALE spaces with A_n singularities and type IIB on a class of singular Calabi-Yau threefolds. The class we study includes as a special case the conifold singularity in which case the perturbative spectrum of the D-string includes the expected massless hypermultiplet with degeneracy one.

1. Introduction

One of the major recent discoveries in string theory has been the appreciation of the existence and importance of solitonic objects [1] [2] [3] [4]. In particular it has become clear that in the case of type IIA compactifications on $K3$ when we have a vanishing 2-cycle we get enhanced gauge symmetries. In case of type IIB compactification on Calabi-Yau threefolds vanishing three-cycles have been proposed to lead to massless charged matter. The basic mechanism suggested [3] is the appearance of solitons corresponding to wrapping of p -branes around vanishing cycles. The exact conformal theory describing the coupling of such solitons to string theory is recently proposed in [4] as D-branes. Even with such a nice picture emerging, we still lack a perturbation scheme in which the wrapped p -branes appear as fundamental string states specifically for $p > 1$, which is the case typically encountered. In this paper we will argue that, at least for all the cases encountered thus far, by using T -duality (which leads to backgrounds with H -fields turned on) the value of p can be changed to $p = 1$. Since the $p = 1$ D-branes are simply the D-strings, dual to type IIB string under strong/weak duality, the nearly massless states are expected to show up in the perturbative spectrum of D-string propagating on the dual compactification which we will call the ‘D-manifold’. ‘D-manifolds’ are manifolds with D-brane configurations (‘skeletons’). We will find that this is indeed the case.

The main motivation behind this work was to better understand the appearance of massless modes in type II string compactifications on Calabi-Yau manifolds. We will first consider type IIA compactification on $K3$ and study how the gauge symmetry enhancement is expected to arise. We find similarity with how the gauge symmetry is enhanced on the D -brane worldsheet through the appearance of massless states in the open string sector when D -branes approach each other [5] [6] [7]. This similarity is not an accident and we argue that 10 dimensional strong/weak duality of type IIB maps one to another. In particular combining type IIB strong/weak duality with the fact that coincident D -branes naturally give rise to enhanced gauge symmetry through standard Chan-Paton factors *explains* the appearance of gauge symmetry for special moduli of type IIA compactification on $K3$. More precisely we will find that type IIA on ALE space with A_{n-1} singularity is equivalent to D-strings on a D-manifold which contains n parallel, coincident Dirichlet 5-branes.

We then consider the compactifications of type IIB on singular Calabi-Yau threefolds. The simplest type is where the Calabi-Yau develops a conifold singularity for which there

is a prediction [3] for the appearance of a massless hypermultiplet. To gain a better insight it is natural to consider more general singularity types. One class is suggested by the observations in [8] that the conifold singularities are topologically equivalent to $c = 1$ strings at the self-dual radius coupled to gravity (by making use of the identification of $c = 1$ at the self-dual radius with the twisted Kazama-Suzuki coset $SL(2)/U(1)$ at $k = 3$ [9]). One evidence in favor of this was the fact that the coordinate ring of the conifold agrees with the ground ring of the $c = 1$ at the self-dual radius [10]¹. Given the rich physical content of $c = 1$ theories and its direct implications for physics of type II compactifications on Calabi-Yau manifolds, it is natural to wonder what type of physics emerges when we consider compactification on Calabi-Yau manifolds which have a singularity corresponding to ground ring of other $d = 2$ strings, classified by *ADE*.

We will mainly concentrate on the A-type $d = 2$ strings, i.e. when the Calabi-Yau develops a singularity corresponding to the ground ring of $2d$ string for $c = 1$ at n -times the self-dual radius [14]. We find that for n -times the self-dual radius there is a $U(n)$ enhanced gauge symmetry as well as matter in the fundamental and adjoint representations of $U(n)$. The mechanism for this rich structure of massless states is having different types of vanishing three cycles where the three cycle collapses to a point or to a curve (this point has also been noted independently in [15]).

Following the connection with D-branes in the case of $K3$ compactifications we find the D -string duals for the singular limits of the corresponding Calabi-Yau. In particular the relevant D-manifold consists of $n + 1$ 5-branes, where n of them are parallel to one another and intersect the other 5-brane on the 3+1-dimensional space-time. This allows us to have a simple description for the degeneracy of nearly massless hypermultiplets. As a special case we derive the fact that in the case of the conifold singularity there is exactly one massless hypermultiplet, as had been proposed [3].

¹ To relate it to Calabi-Yau compactification, non-critical $c = 1$ strings compute the leading behavior for prepotential and some special amplitudes involving $(2g - 2)$ graviphotons and 2 gravitons [11] [12]. The computation of [13] for such corrections in the presence of a nearly massless hypermultiplet and its agreement with the partition function of $c = 1$ at the self-dual radius provides a strong all genus check for the appearance of the massless hypermultiplet at the conifold points of Calabi-Yau moduli [3].

2. K3 and Gauge Symmetry Enhancement

It is interesting to study in detail how type IIA on $K3$ leads to gauge symmetry enhancement through vanishing 2-cycles. For simplicity we will mainly consider A_{n-1} singularities but make some comments about D and E as well. For our purposes, the *local* analysis near singularity is sufficient. Degenerating family of $K3$'s can locally be modelled by a family of (resolved) A_{n-1} ALE spaces: surfaces given by

$$\prod_{i=1}^n (\zeta - \mu_i) + z^2 + w^2 = 0. \quad (2.1)$$

in \mathbf{C}^3 with coordinates z , w and ζ . This family is parameterized by $\{\mu_i\}_{i=1}^n$ that may serve as local coordinates on moduli space of $K3$. To be precise we also have to specify a point on the Kähler moduli for $K3$. For all $\mu_i \neq \mu_j$, the surface (2.1) is smooth. When μ_i meets μ_j , an A_1 singularity develops and a 2-sphere S_{ij}^2 vanishes. If we take μ_i to be real the vanishing 2-sphere can be viewed as a real section of (2.1). Similarly, when $\mu_{i_1} = \dots = \mu_{i_k}$ the surface (2.1) develops A_{k-1} singularity with $k-1$ independent cycles $S_{i_1 i_2}^2, \dots, S_{i_{k-1} i_k}^2$ shrinking to zero size.

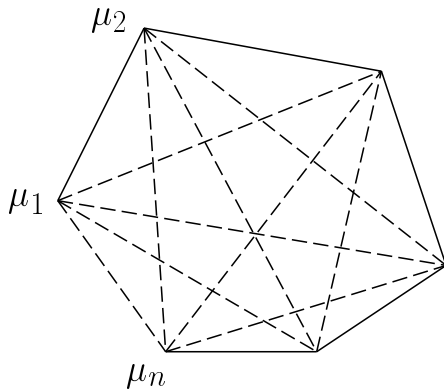


Fig.1

Figure 1 captures the μ -plane. Each line connecting two values of μ_i represents a vanishing S^2 . A sublattice in $H_2(K3)$ generated by all cycles $S_{12}^2, \dots, S_{(n-1)n}^2$ coincides with A_{n-1} root lattice. The monodromy of a cycle S around a singular locus $\mu_i = \mu_j$ is given by Picard-Lefschets formula

$$S \rightarrow S - S_{ij}(S \circ S_{ij}). \quad (2.2)$$

The intersection form on vanishing 2-cycles coincides with Cartan form on A_{n-1} and therefore the monodromy (2.2) is just a Weyl reflection. The whole monodromy group coincides with a Weyl group of A_{n-1} . It is important that taking a vanishing cycle, say S_{12}^2 , and applying the monodromy group to it one obtains *all* other vanishing cycles. In Type IIA compactification on $K3$ extra light multiplets are expected to appear in such a limit [2] provided that the B -field part of the Kähler moduli is suitably chosen [16]. The basic mechanism [3] (suggested in the context of Calabi-Yau threefolds) is that there are nearly massless solitons where appropriate 2-branes wrap around the vanishing 2-cycles. The precise details of how this happens has not been clarified. A promising candidate of the corresponding 2-branes has been found recently [4] to be a D-brane which has a simple conformal theory description.

2.1. *D-branes and Gauge Symmetries*

It is quite striking that the mechanism of enhanced gauge symmetry we observed here is very similar to the mechanism of enhancement of gauge symmetry on the worldsheet of D -branes when they approach each other [5][6][7]. In particular the masses of massive gauge particles are measured by the separation of pairs of points μ_i, μ_j ². In the case of the D -branes, if we have n parallel nearby D -branes, there is an enhancement of gauge symmetry to $U(n)$. The corresponding gauge particles are given by open string states connecting two nearby D -branes. Again the mass of each of these nearly massless gauge bosons is related to the stretching of open strings, or equivalently to how far they are from one another. We will now see that this amazing similarity is not an accident and leads to an unexpected connection between type IIB strong/weak duality in 10 dimensions and gauge symmetry enhancement for type IIA upon compactification on $K3$ with A_{n-1} singularity.

To understand this connection it turns out to be crucial to study the structure of conformal theory near the singularities of $K3$. This has been done recently [17] and it turns out that the conformal theory is exactly solvable for all the ADE singularities of $K3$. What is found is that for the A_{n-1} case it is (a capped version of) the same conformal theory as that of symmetric 5-branes [18] with H charge n . A connection between the conformal theory associated to A_1 singularity and symmetric fivebrane had been anticipated [19].

² Actually note the the Kähler moduli should also be added to the μ_i to give a correct measure of mass as explained before.

However a crucial subtlety appears leading to an exchange between type IIA and type IIB: It turns out that type IIA on ALE space with A_{n-1} singularity is equivalent to type IIB on symmetric fivebranes of H -charge n . The basic idea is that if we think of $K3$ as fibered over the sphere with tori as the fiber (where the torus becomes singular at the analog of $\zeta = \mu_i$) the $R \rightarrow 1/R$ duality transformation of the fiber (mapping type IIA to type IIB) implies that points where the torus degenerates are also the points where there is an H charge [20].

In this context we thus expect that type IIB on symmetric fivebranes leads to enhanced gauge symmetries in six dimensions. What would be a candidate for the corresponding soliton? Being the dual to type IIA description it can be either a 1-brane or a 3-brane. We will now see that they are indeed 1-branes as is suggested by Figure 1.

The conjectured $SL(2, Z)$ duality of type IIB in 10 dimensions [1] has been recently analyzed in connection with the prediction of new types of strings [21] and in connection with the proposal of Polchinski [4] in constructing bound states for D-branes [6]. The strong/weak duality of type IIB in 10 dimensions maps the B field of the NS-NS sector to the \tilde{B} field of the R-R sector. Strings are the source of B and Dirichlet 1-branes, the D-strings, are the source of \tilde{B} . Thus strong/weak duality maps type IIB strings to D-strings. To search for 1-brane solitons, which are also the same as D-strings, all we need to do is to study D-strings perturbatively where the solitons would appear as elementary D-string states. We have to construct what the background looks like for the D-strings. Since the symmetric five branes are the source for the field which is spacetime dual³ to B and Dirichlet 5-branes are the source for the spacetime dual to \tilde{B} field, we see that strong/weak duality should exchange the symmetric 5-branes with the Dirichlet 5-branes. In other words type IIB strings on a manifold consisting of n -symmetric 5-branes is dual to D-strings on a D-manifold consisting of n parallel Dirichlet 5-branes.

On the other hand type IIB on a manifold with a symmetric 5-branes with H charge equal to n is equivalent to type IIA on a space with A_{n-1} singularity [17]. Note that on the 6 dimensional worldsheet of each of the Dirichlet 5-branes we have a $U(1)$ gauge symmetry. Moreover we expect that when n Dirichlet 5-branes approach each other we obtain, through the standard Chan-Paton factors, an enhancement of gauge symmetry from $U(1)^n \rightarrow U(n)$, i.e. a $U(n)$ gauge symmetry in 6 dimensions. This gauge symmetry enhancement is exactly what we would expect for type IIA on ALE spaces with A_{n-1} singularity! (the extra $U(1)$ is

³ The duality here means that the field strengths are Poincare dual.

related to the $U(1)$ gauge field of type IIA in 10 dimensions). We have thus connected type IIA enhancement of gauge symmetry to type IIB strong/weak duality in 10 dimensions through standard Chan-Paton mechanism. Moreover the solitonic Dirichlet 2-branes of type IIA which are becoming massless are mapped to the nearly massless open string states of D-string (depicted by lines connecting pairs of μ_i in Figure 1). Note also that the fact that we have to adjust *four* real quantities to get a massless states is consistent and even more clear in the D-string description.

2.2. Type IIB on $K3 \times T^2$

In preparation for our discussion of the Calabi-Yau threefolds we wish to discuss type IIB on $K3 \times T^2$ (changing from our starting point which was type IIA on $K3$). Type IIB compactification on $K3 \times T^2$ is equivalent to type IIA compactification on $K3 \times \tilde{T}^2$ where \tilde{T}^2 is the torus dual to T^2 with Kähler and complex moduli exchanged. We can now have a different perspective about the appearance of massless particles when we consider $K3$ with vanishing 2-cycles [19]: The extra massless gauge particles required by string duality can be understood from the view point of the vanishing 3-cycles. The geometry of vanishing cycles V_{ij} in this case is $S_{ij}^2 \times S^1$, where the S_{ij}^2 comes from the singular locus on $K3$, where μ_i meets μ_j , and S^1 is one of the cycles of the T^2 . Cartan subalgebra of $SU(n)$ corresponds to complex moduli $\mu_i - \mu_{i+1}$, while the nilpotent subalgebras \mathcal{N}_\pm are spanned by solitonic configurations – three-branes wrapped around vanishing cycles. Primitive cycles V_{ii+1} correspond to simple roots (solid lines on Figure 1), while V_{ij} are in one to one correspondence with positive non simple root vectors (dashed lines on Figure 1). Negative roots correspond to three-brane configuration with reversed orientation.

Let us discuss the monodromy group for $K3 \times T^2$. Since we do not vary a modulus of T^2 , the parameter space is the same as for $K3$. Thus the monodromy group is the Weyl group of A_{n-1} again. An action of the Weyl group on 3-cycles is induced by its action on 2-cycles of $K3$. In particular, if we denote by A and B two circles of T^2 , the cycles $\{S_{ij}^2 \times A\}$ are never mixed with the cycles $\{S_{ij}^2 \times B\}$. Moreover, cycles from the same group do not intersect⁴: $(S_{ij}^2 \times A) \circ (S_{kl}^2 \times A) = 0$. In the context of Type IIB compactification this means that in 4 dimensions, the particles corresponding to $\{S_{ij}^2 \times A\}$ are mutually local. On the other hand, since $S_{ij}^2 \times A$ intersect $S_{kl}^2 \times B$ the full set of 4-dimensional

⁴ The fact that there is still a nontrivial monodromy does not contradict Picard-Lefschets simply because the setup is different.

particles is not mutually local. The U -duality transformation of T^2 exchanges A and B , making electric-magnetic duality transparent [6].

The hypothesis is that the solitonic configurations we just described are the only stable ones. This partially follows from transitivity of monodromy group as it acts on vanishing cycles. Indeed, applying monodromies to a stable solitonic configuration corresponding to V_{12} one gets *all* other solitons provided one does not encounter jumping phenomena, which is guaranteed by $N = 4$ supersymmetry. Note that since we have $N = 4$ in $d = 4$ we should expect in the $N = 2$ terminology one massless vector multiplet and one massless hypermultiplet both in the adjoint of $SU(n)$.

It should be noted that we should tune the Kähler moduli of vanishing two cycles in an appropriate way. Note in particular that the moduli space of $K3$ does not naturally split to Kähler and complex deformations. This fact is going to be implicit in our discussions of Calabi-Yau singularities below and will not be mentioned further.

3. Singular CY's, $c = 1$ String and $U(n)$ Gauge Symmetry

Let us summarize what we have learned from the example with $K3 \times T^2$. The whole analysis of singularity was purely local. We never used the global structure of $K3$ except for the $N = 4$ structure. The only relevant piece of geometry of T^2 was the existence of a circle S^1 . Since the monodromy never mixes the A and B cycles one can discuss them independently.

Now consider a family of 3-dimensional Calabi-Yau spaces. It is natural to ask what happens if we have different types of vanishing 3-cycles. The 3-cycle can collapse to **A**) a point, **B**) a line, or **C**) a surface. Case **A** is encountered in the conifold singularity of a Calabi-Yau represented locally by

$$x^2 + y^2 + z^2 + w^2 = \mu \tag{3.1}$$

Note that the 3-cycle can be viewed as a real section of the above (take μ to be real and positive) in which case it defines S^3 . Note that as $\mu \rightarrow 0$ the whole sphere shrinks to a point. Compactifying Type IIB string on such Calabi-Yau one expects to end up with one hypermultiplet of mass $\sim \mu$ [3].

For the example of case **B** one can consider a family of $K3 \times T^2$, discussed above, where $K3$ develops a singularity by shrinking of several 2-spheres S_i^2 to zero size.

For examples of type **C** where the vanishing cycle collapses to a surface, the natural candidate is $T^2 \times T^4$ compactification where the modulus of T^2 goes to infinity—however this is infinitely far away in the moduli of Calabi-Yau. In fact it is probably the case that the singularities of type **C** do not occur at finite distance in the moduli of Calabi-Yau.

Locality leads us to a natural hypothesis that *whenever several 3-cycles shrink to a circle S^1 and locally the geometry is $\text{ALE} \times S^1 \times \mathbf{R}$, gauge symmetry appears and we find two $N = 2$ multiplets, a vector multiplet and a hypermultiplet both in the adjoint representation of the gauge group.* This is the field content of $N = 4$ Yang-Mills theory. In general we do not expect to have $N = 4$ supersymmetry for compactifications on CY. Therefore we may expect to find other $N = 2$ hypermultiplets charged with respect to the gauge group which break $N = 4$ down to $N = 2$. Exactly this happens in the examples below.

Finally, it seems that all we need to obtain a nonabelian gauge symmetry in Type IIB compactification is a Calabi-Yau manifold with a singularity along a complex curve C . In the vicinity of C the Calabi-Yau should look like a fibration by ALE spaces. Also, the curve C should have at least one noncontractible cycle (for example, $C = \mathbf{C}^* \sim S^1 \times \mathbf{R}$ is good enough).

Before embarking on examples of such singularities, let us recall that the conifold singularity is related to the $c = 1$ string at the self-dual radius. It is natural to wonder what type of singularity one obtains when one considers n times the self-dual radius. The corresponding ground ring has been computed [14] at $\mu = 0$ to be (by a change of coordinates)

$$(x^2 + y^2)^n + z^2 + w^2 = 0 \quad (3.2)$$

This singularity has a structure similar to conifold and (2.1) singularity, namely if we take for instance $n = 1$ it is of the same type as **A** and if we take $\zeta = x^2 + y^2$ it is of the same type as **B**.

Let us consider the perturbation of (3.2) given by

$$W = \prod_{i=1}^n (x^2 + y^2 - \mu_i) + z^2 + w^2 = \prod_{i=1}^n (\zeta - \mu_i) + z^2 + w^2 = 0 \quad (3.3)$$

Note that the cosmological constant deformation of $c = 1$ model at n -times the self-dual radius corresponds to all $\mu_i = \mu$. This more general deformation by arbitrary μ_i seems to correspond to $n - 1$ discrete states with zero momentum (with the dressing, violating Seiberg's condition) [22].

For $\mu_i = \mu$ the manifold (3.3) is singular along the curve

$$(x^2 + y^2 - \mu)^{n-1} = 0, \quad z = 0, \quad w = 0. \quad (3.4)$$

The transversal fiber is the A_{n-1} ALE space. Locally, around the singular locus the manifold looks like an ALE fibration over $S^1 \times \mathbf{R}$. We expect that this singularity corresponds to enhanced $SU(n)$ gauge symmetry. We will see below that $N = 4$ is broken to $N = 2$ by extra matter in fundamental representation.

For generic μ_i it is easy to see that as long as none of the $\mu_i = 0$ and $\mu_i \neq \mu_j$ for all $i \neq j$ the non-compact threefold defined locally by (3.3) is non-singular. The dimensions of $H^{1,1}$ and $H^{2,1}$ are given by $h^{1,1} = n - 1$ and $h^{2,1} = n$ which means that the corresponding type IIB theory will have $n - 1$ hypermultiplets (in addition to the dilaton) and n vector multiplets. Let us first understand why $h^{2,1} = n$: The easiest way is to note that we have n parameters μ_i which deform the theory away from the singularity and they are in one to one correspondence with elements of $H^{2,1}$. Moreover we can give an explicit basis for the corresponding 3-cycles as follows. Consider $\mu_i \rightarrow 0$ and fix all the other μ_j 's at generic points. Then it is easy to see by examining the defining equation that we end up getting a conifold singularity, i.e. a vanishing 3-cycle S_i^3 . In this way we can use $V_i = [S_i^3]$ to form a basis for the vanishing 3-cycles. Moreover according to [3] for each $\mu_i \rightarrow 0$ we should get a massless hypermultiplet charged under the $U(1)$ whose scalar component corresponds to μ_i . To describe $H^{1,1}$ note that we have a projection from our threefold to a twofold given by sending $(x, y, z, w) \rightarrow (\zeta = x^2 + y^2, z, w)$ and we can pull back the corresponding $n - 1$ elements of $h^{1,1}$ which are dual to the $n - 1$ vanishing 2-cycles of H_2 . It is not difficult to show that there are no other elements of H^2 and H^3 with compact support.

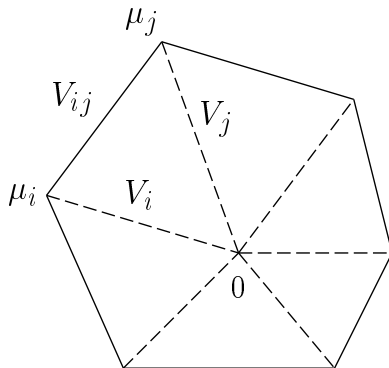


Fig.2

Suppose we take generic μ_i except for letting two of them approach each other. Then not only we get a vanishing two cycle, but also we get a vanishing 3-cycle: It suffices to consider the case $n = 2$, and take

$$W = (x^2 + y^2 - R^2)^2 + z^2 + w^2 - r^2 = 0$$

with $R \gg r$ real and positive. Then r measures the separation of the two μ_i and R measures their distance to the origin. Then if we consider the above equation over the reals we see that it has the topology of $S^2 \times S^1$, where r measures the radius of S^2 and R measures the radius of S^1 (more precisely the radius of S^1 varies from $R + \sqrt{r}$ to $R - \sqrt{r}$). This implies that as $r \rightarrow 0$ we have a vanishing 3-cycle of type B discussed above. It is easy to see that this 3-cycle is not independent of V_i defined above. In fact, if $\mu_i \rightarrow \mu_j$ the vanishing 3-cycle V_{ij} which we realized as an $S^2 \times S^1$ is homologically equivalent to

$$V_{ij} = V_i - V_j$$

If the local analysis of the string duality is any hint, we can expect that if two μ_i approach each other the $U(1) \times U(1)$ related to the two μ 's get enhanced to $U(1) \times SU(2) \sim U(2)$ vector multiplet with an extra adjoint hypermultiplet of $SU(2)$. Moreover, there are also two massive hypermultiplets that form a doublet of $U(2)$ (both of them become massless as μ_i and μ_j approach zero). The mass is related to the fact the $\mu_i \neq 0$ and thus we are in the Coulomb phase of the diagonal $U(1)$ which couples to the fields in the fundamental representation. Note that this is consistent with dynamics of gauge fields: in the infrared we can ignore massive fields, in which case the theory behaves as if it was an $N = 4$ theory and we thus expect to have an exact $U(2)$ symmetry even in the quantum theory. Note that this is needed in order for the above picture to make sense because the singularity of the Calabi-Yau manifolds must have a direct bearing on the exact, quantum corrected, physics. In particular if we did not have the adjoint of the gauge group we know that the $SU(2)$ symmetry disappears from the quantum moduli and we only realize a $U(1)$ gauge symmetry in the infrared, in contradiction with the geometry of the Calabi-Yau.

To understand the full matter content of this theory consider the action of the monodromy group on 3-cycles H_3 . For A_{n-1} case the monodromy group coincides with group S_n of permutation from n elements. It acts on cycles V_{ij} and V_i in an obvious way by permuting the indices. Cycles V_i correspond to the matter sector. Assuming that there is a stable 3-brane configuration wrapped around one cycle one immediately concludes that

3-branes wrapped around all V_i are stable solitons (at least in some limit when all $\mu_i \rightarrow 0$). The set of particles corresponding to V_i should form the fundamental representation of $U(n)$.

To summarize, we are suggesting that if $\mu_i = \mu, (i = 1, \dots, n)$ the field theory dynamics is the same as for an $N = 2$ $U(n)$ gauge theory with a massless adjoint hypermultiplet of $SU(n)$ and a massive fundamental of $U(n)$ whose mass is related to μ . As we move μ_i to generic points the theory goes to the Coulomb phase for which the surviving gauge group is $U(1)^n$. Note also that if we took into account the graviphoton the gauge group is actually $U(n) \times U(1)$.

The parameters μ_i are related to the condensates of the scalar field ϕ of the vector multiplet as follows $\det(\phi - \zeta) = \prod(\mu_i - \zeta)$. The fundamental field is always massive. Of the adjoint hypermultiplet of $SU(n)$, $n - 1$ of them from Cartan remain massless and the rest pick up mass. Thus at generic μ_i the massless (field theory) degrees of freedom are n vector multiplets corresponding to $U(1)^n$ and $n - 1$ neutral hypermultiplets⁵. This is as we expected because $h^{1,1} = n - 1, h^{2,1} = n$. Since the infrared dynamics for $\mu \neq 0$ is that of $N = 4$ $U(n)$ theory, the gauge coupling is not quantum corrected. This is in line with the fact that the prepotential in this theory is expected to be

$$F_0 = \sum_i \frac{1}{2} \mu_i^2 \log \mu_i \quad (3.5)$$

Note that if we set $\mu_i = \mu$ we get

$$F_0 = \frac{n}{2} \mu^2 \log \mu$$

This is consistent with the fact that at $R = n$ the modified coupling constant $n\mu^2 \rightarrow \mu^2$ is the one which does not transform with $R \rightarrow 1/R$ duality, and is to be identified with the $c = 1$ cosmological constant.

Let us discuss a more complicated example with a gauge group $U(n) \times U(m)$ and massive matter in a (n, m) representation. Consider a hypersurface given by equation

$$\prod_{i=1}^n (x^2 + y^2 + \mu_i) + \prod_{j=1}^m (z^2 + w^2 + \nu_j) = M \quad (3.6)$$

⁵ It is also possible to give vev to a cartan of the hypermultiplet. The gauge symmetry is again broken to $U(1)^n$, but now the fundamentals are still massless because they don't couple to the hypermultiplet.

For $m = 1$, this reduces to the previous case. To see $U(n) \times U(m)$ symmetry, take $\mu_i = \mu$, $i = 1, \dots, n$, $\nu_j = \nu$, $j = 1, \dots, m$ and $\mu^n = \nu^m = M$. Then there is an A_{n-1} singularity along the curve \mathcal{C}_1 : $\{x^2 + y^2 + \mu = 0, z = w = 0\}$ and an A_{m-1} singularity along the curve \mathcal{C}_2 : $\{x = y = 0, z^2 + w^2 + \nu = 0\}$. To see nm 3-cycles corresponding to a (n, m) hypermultiplet, one takes $\mu_i \rightarrow 0$, $\nu_j \rightarrow 0$ for every pair (i, j) . For (x, y, z, w) near origin, (3.6) reduces to a resolved conifold:

$$A(x^2 + y^2) + B(z^2 + w^2) = M$$

which has a vanishing 3-sphere. The mass of the corresponding 3-brane soliton is governed by M .

A compact version of this example can be given using double elliptic fibrations over \mathbf{P}^1 , discussed in [23] and [24]. Namely, consider a complete intersection in $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^1$ given by two equations

$$\begin{aligned} P(x)z_1 + Q(x)z_2 &= 0 \\ R(y)z_1 + S(y)z_2 &= 0 \end{aligned} \tag{3.7}$$

where $(z_1 : z_2)$ are homogeneous coordinates of \mathbf{P}^1 , x and y are sets of coordinates of two \mathbf{P}^2 's and P, Q, R, S are four homogeneous polynomials of degree 3. Obviously, this complete intersection is a Calabi-Yau three-fold, smooth for generic P, Q, R, S .

We can think of (3.7) as a fibration over \mathbf{P}^1 . Fixing a point $(z_1 : z_2) \in \mathbf{P}^1$ one sees that a fiber is a product of two tori — cubics in \mathbf{P}^2 's given by the first and the second equations in (3.7) respectively. These two tori are completely independent. What 2-manifold does one get if one takes a fibration over \mathbf{P}^1 by just one torus?

Examining this closer, one sees that an equation $P(x)z_1 + Q(x)z_2 = 0$ defines an almost del Pezzo surface — a blowup of \mathbf{P}^2 in nine (special) points. Indeed, when at least one of P, Q is not 0, we can find a single point in \mathbf{P}^1 and when both $P = Q = 0$ which happens in 3×3 points on \mathbf{P}^2 , any $(z_1 : z_2)$ solves the equation. This almost del Pezzo surface has $h^{1,1} = 10$ and Euler characteristic 12. When we think of it as a toric fibration over \mathbf{P}^1 it means that generically there are 12 singular fibers where the torus degenerates. When two of these fibers collide, almost del Pezzo surface develops a nodal A_1 singularity.

Let us return to the three-fold (3.7). To find its Hodge numbers we will employ two complimentary pictures where (3.7) is a toric fibration over either of two almost del Pezzo surfaces given by the first or the second equations of (3.7) respectively. Taking pullback of $(1, 1)$ -forms from a surface to (3.7) one gets 10 $(1, 1)$ -forms one of which is a Kähler class of

\mathbf{P}^1 . Thus taking pullbacks from both surfaces one gets $10+9 = 19$ different $(1,1)$ -forms, so $h^{1,1} = 19$. Since the Euler characteristic is zero, $h^{2,1} = 19$. All the complex deformations are the monomial deformations of (3.7).

Now let us consider the picture of (3.7) as a double toric fibration over \mathbf{P}^1 . On \mathbf{P}^1 there are 24 special points divided into two sets of 12 points. Over each point from the first (the second) set the first (the second) torus degenerates. (It does not lead though, to a degeneration of (3.7) itself.) Changing coefficients in (3.7) one can move two special points on top of one another. Suppose *both* these points belong to *the first set*, then there develops an A_1 singularity along the second torus. Indeed, almost del Pezzo surface corresponding to fibration over \mathbf{P}^1 by the first torus will have a node in this situation, and considering (3.7) as a fibration over that del Pezzo by the second torus one verifies the claim.

Now suppose that we move *one* point from *the first set* and *another* from *the second set* on top of each other. Then both tori degenerate over this point. It is not difficult to see that this corresponds to a conifold singularity of the three-fold (3.7). Indeed, near singularity each torus looks like a conic: $x_1x_2 = z$ and $y_1y_2 = z$, where z is a coordinate on \mathbf{P}^1 . Combining these two equations one gets the conifold equation:

$$x_1x_2 - y_1y_2 = 0.$$

In general, m points from the first set and n points from the second set coincide. Then there is a curve (torus) with A_{m-1} singularity along it and another curve, also a torus, with A_{n-1} singularity. Obviously, $m+n \leq 20$ and $0 \leq m, n \leq 12$ in this example.

It will be very interesting to generalize the machinery discussed above for D_n singularities which correspond to $SO(2n)$ gauge groups ⁶.

3.1. D-brane Interpretation

Given the fact that we found a dual D-string description for the appearance of gauge fields for singular *ALE* spaces with A_n singularity, it is natural to ask whether there is any D-manifold on which the light spectrum of D-string configuration matches the above spectrum. We will first consider the singularity corresponding to n -times the self-dual radius and propose a corresponding D -manifold. We will then generalize it to the $U(n) \times U(m)$ considered at the end of previous section. Then we will show *why* the proposed

⁶ The perturbed ground ring for $c = 1$ is given by $W = y^2 + (z + w^2 - \mu)(z^{n-2} + x^2)$.

D -manifold is indeed the dual of the type IIB on the corresponding singular Calabi-Yau threefold.

From the description given above it is clear that we need an additional object corresponding to the point $\mu = 0$ of Figure 2, and moreover we expect that the ‘lines’ connecting μ_i to 0 should now correspond to hypermultiplets rather than vector multiplets. Moreover we need fewer supersymmetries, i.e., $N = 2$ in 4 dimensions. Thus we should be looking for a D -manifold consisting of $(n + 1)$ D -brane skeletons, n of which are of one type and one is of a different type. Consider the n Dirichlet branes of the first type, which we call the D_1 type given by restricting

$$(x_6, x_7, x_8, x_9) = v_i$$

for $i = 1, \dots, n$ where spacetime coordinates run from x_0, \dots, x_9 . The rest of the coordinates satisfy Neumann boundary conditions. The other D -brane should not be parallel to this, otherwise we would get $N = 4$ supersymmetry rather than $N = 2$. Moreover we want to get four dimensional Poincare invariance. So we consider the D_2 type to be

$$(x_4, x_5, x_6, x_7) = (0, 0, 0, 0)$$

(with no loss of generality we have fixed the only D_2 brane to have fixed boundaries at the origin of the coordinates). The D -manifold consists of n 5-branes of one type and one 5-brane of another type which intersect on a $3 + 1$ -dimensional spacetime if any of the $v_i = 0$. It is now easy to see how we get the $SU(n)$ gauge symmetry and the adjoint matter: They simply come from the open string sectors connecting the D_1 branes to each other. Note that the adjoint matter simply comes from dimensional reduction from 6 to 4, as we expected. The $U(1)$ gauge symmetry comes from the open string connecting D_2 to itself. Now consider open strings connecting any of the D_1 branes to the D_2 brane. It involves boundary conditions on both sides which are DD, NN or DN (where N stands for Neumann and D for Dirichlet). Then the bosonic oscillators are integral in the DD and NN directions but half-integral in the DN directions. Thus in the R sector (NS sector) the NSR fermions are integral (half-integral) in the DD and NN directions but half-integral in the DN directions (note that the supercurrent has integral (half-integral) expansion in the R sector (NS sector)). Looking at the two types of D -branes we see that in the light-cone ($x_0 = \tau, x_1 = \sigma$) the directions x_2, x_3 are NN type, x_6, x_7 are DD type and x_4, x_5, x_8, x_9 are DN type. We thus have the same structure as appears in the right-moving twisted sector of

heterotic strings upon compactification on $K3 = T^4/Z_2$. We thus get a system with $N = 1$ in six dimensions or $N = 2$ in four dimensions which leads to one hypermultiplet (to see that is a hypermultiplet it is sufficient to study the NS sector where one finds four scalar degrees of freedom). Taking into account the Chan-Paton factors at the open string ends we see that the hypermultiplet is charged and that it is in the fundamental representation of $U(n)$ exactly as we were expecting! Note also that the mass of the hypermultiplet goes to zero if the 6, 7 components of the corresponding v_i approach zero (i.e. we have to tune two parameters to get a massless state, as expected for the conifold).

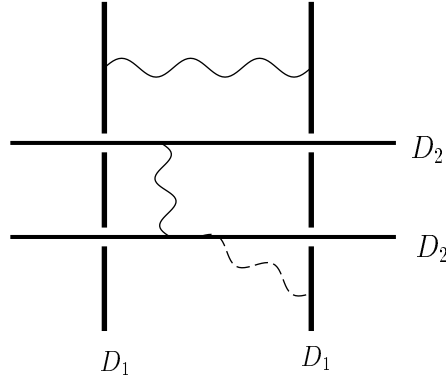


Fig.3

It is now clear how to generalize it to the case of $U(n) \times U(m)$ considered at the end of last section. In that case we will naturally consider n D_1 branes and m D_2 branes. It is clear that we will obtain the matter spectrum in the (n, m) representation coming from the Chan-Paton factors lying on each of the two D-branes.

So far we have only proposed a D-string vacuum on a D-manifold which leads to the same spectrum as type IIB on the corresponding singular Calabi-Yau. Can we actually derive this from the type IIB $SL(2, Z)$ duality in 10 dimensions just as we did for gauge symmetry enhancement? Note in particular that if we understand the case of $n = m = 1$ the rest will follow in an obvious way by ‘superposition’. The case $n = m = 1$ is the ordinary conifold singularity given by

$$xy - zw = \mu$$

Let us introduce another variable ζ and write the conifold equivalently as

$$xy = \zeta$$

$$zw = \zeta - \mu$$

The H-version of the ‘stringy cosmic string’ picture thus gives us one H string, whose projection is at $\zeta = 0$ and another one whose projection is at $\zeta = \mu$. The corresponding fibers $xy = \text{const}$ and $zw = \text{const}$ are independent of each other. Thus applying the duality argument for each one, sketched before we obtain two 5-branes, which we can easily identify with D_1 and D_2 described above! Note that in this correspondence we had to apply $R \rightarrow 1/R$ duality twice, once for each fiber; thus type IIB on the conifold is equivalent to type IIB again, but now with suitable H -fields turned on. Applying $R \rightarrow 1/R$ twice has shifted the value of $p = 3$ for the solitonic p -brane by two units down to $p = 1$, and the soliton is now realized as an elementary D -string state.

Note in particular that we have shown that the existence and the degeneracy of the solitonic massless hypermultiplet for type IIB strings on a conifold is a consequence of the conjectured $SL(2, \mathbf{Z})$ duality of type IIB in 10 dimensions and is in agreement with the conjecture of Strominger [3].

4. Final Remarks and Conclusions.

We have seen that the appearance of massless states in type II compactifications follows from $SL(2, \mathbf{Z})$ duality of type IIB in ten dimensions. It can be described perturbatively using D-strings propagating on suitable D-manifolds. Moreover they are dual to type IIB in the presence of D-branes and the massless states (be they gauge bosons or massless charged matter) can be viewed as light states in an open string subsector of the D-string in the presence of D-branes.

We described the appearance of A_n gauge symmetry through Chan-Paton factors of the open string sector of D-string. We could have talked about the D and E series, in which case what is most likely to happen is that not all the configurations of open strings between the D-branes are stable (similarly to the story of solitons of $N = 2$ Landau-Ginzburg theories in 2 dimensions [25]), and the stable ones reproduce the expected root lattice of the group. This deserves further study.

So far we have talked about non-compact D-manifolds. This was because that was sufficient to describe the dual to the local singularity of Calabi-Yau. However if we are to consider the dual to the compact Calabi-Yau, the D-manifold will have to be compact. Once we do so we have to develop the notion of D-manifolds rather carefully. There can be various type of restrictions. In particular if we take the original type IIB compactification

with H -field turned on, the condition of conformal invariance and the value of central charge puts restrictions. Thus taking its dual we will have dual restrictions for the D-manifold. Say if we begin compactifying the space on a two dimensional torus, and consider D 5-branes whose Dirichlet boundaries partially reside on the torus the story is going to be changed. In fact by considering its dual which is the H dual of the ‘stringy cosmic strings’, the restriction on the number of cosmic strings being less than 24 [20] puts a bound on the number of 5-branes (note also that this implies that their positions are not independent — a key point which has to be studied further).

The relation between singularities of Calabi-Yau threefolds and non-critical string theory is still quite mysterious. In particular it would be important to explain the partition function of non-critical strings in terms of physical spectrum of Calabi-Yau compactification. For example it would be interesting to check the correction to F_1 which is expected to be

$$F_1 = \frac{-1}{24}(n + \frac{1}{n})\log\mu$$

and compute the correction to R^2 [11][12]. Note that unlike the case considered in [26] as $\mu \rightarrow 0$, we have a strongly interacting $U(n)$ theory, for which the computation of R^2 may require understanding some subtleties. Also of interest would be to study realizations of the $R = n$ singularities in compact Calabi-Yau manifolds. These issues are currently under investigation⁷

Another interesting direction to pursue is the duality between type IIA on $K3$ with type I on T^4 . It is likely that the D-string formulation of type IIA on $K3$ is very close to the type I compactification on T^4 and it is likely to be a useful set up in understanding this duality better.

Finally we would like to make a remark in connection with the stringy realization of dualities in the $N = 1$ supersymmetric gauge theories (see [29] for a review). One of the simplest examples, which is accidentally an $N = 2$ system, is an electric $SO(3)$ gauge theory with a triplet matter which is dual to a magnetic theory of $SO(2)$ with a doublet and a neutral meson. This is in fact the same as the $N = 2$ duality of Seiberg and Witten [30], where the electric theory is equivalent to $N = 2$ theory of $SU(2)$ Yang-Mills and the magnetic theory is equivalent to the $U(1)$ theory with a massless hypermultiplet

⁷ A promising candidate for this is the strong coupling locus $y = 1$ appearing in the examples of $N = 2$ duality considered in [27]. This is currently being investigated [28] with results similar to what one expects from the picture presented here.

corresponding to the massless magnetic monopole of the $SU(2)$ electric theory. One may at first think that this duality has already been discovered in string theory [27] where the electric side is the heterotic one and the magnetic side is the type II side. However that would not be quite accurate: the magnetic side contains *perturbative* states which are charged under the $SO(2)$ whereas in the type II case the charged hypermultiplets are still 3-brane solitonic states! What we have found here is that we should expect the magnetic side in the string theory to be realized as a D-string on a D-manifold where the massless $U(1)$ charged states are given by perturbative states. One should try to construct the D-manifold equivalents of *compact* Calabi-Yau compactifications to realize this idea more concretely. In fact since the examples of [27] are $K3$ fibrations [31] the dual D-manifold can easily be described as a similar fibration with the fibers replaced by the appropriate configuration of D 5-branes.

Since we have now realized a duality in string theory in the same sense as duality in supersymmetric field theories one may hope to obtain more general classes by studying D-strings on D-manifolds which leave only $N = 1$ supersymmetry in four dimensions. It is likely that the duality between D-strings and heterotic strings in the $N = 2$ case generalizes to interesting $N = 1$ dualities.

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